What Do You Expect to Happen When ...?

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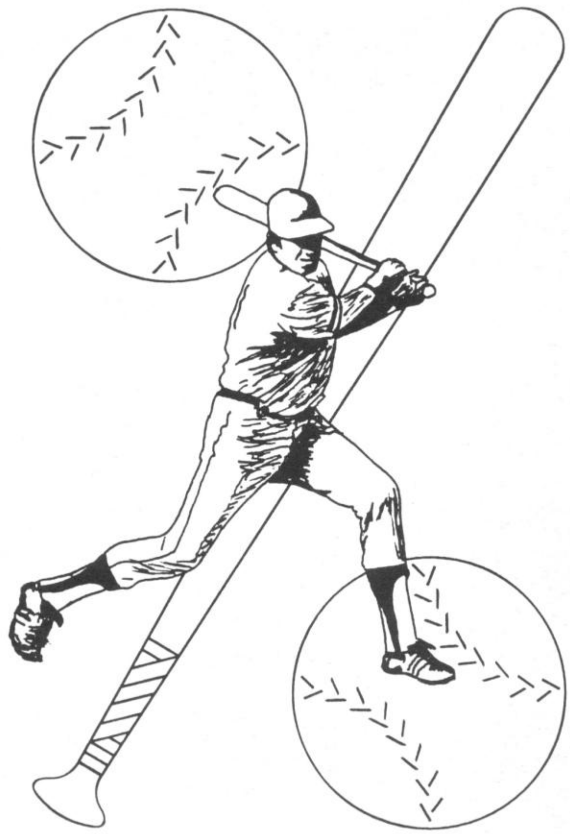


What do expect to happen when the weather report calls for an 80% chance of rain for tomorrow? What do you expect to happen when your favorite major league player, who has a batting average of .300, comes to bat, or when you buy a raffle ticket for $1 in an attempt to win a prize worth $800? These questions can be answered mathematically using the concept of expected value and information regarding the probability or the percent chance of an event occurring.

A 50% chance of rain is predicted when the weather conditions are such that in the past it has rained on 5 out of 10 days which have had similar conditions [Travers, et. al. 1985].

Example #1: Suppose that Atlanta, Memphis, New Orleans, and St. Louis all have weather predictions of a 50% chance of rain on a certain day in May. In how many of these cities would you expect it to rain?

Solution: There are four cities, each with a 50% chance of rain, or a probability of 5/10 of rain. Multiply the probability of rain times the number of cities to find the number in which we would expect it to rain. (50%)(4) = (5/10)(4) = 2. We would expect rain in two of the four cities.

You Try It #1: During a very gloomy October in Chicago, predictions of a 60% chance of rain were made on each of 20 days. How many of the 20 days with this weather forecast would you expect it to rain?

Example #2: Suppose your favorite major league player has a batting average of .300. This means that he has averaged three hits in every 10 official times at bat since .300 = 3/10, and we may expect him to perform at approximately the same level in the future. Suppose that he will come up to bat 20 times in the next five games. How many hits do we expect him to make?

Solution: Multiplying .300 by 20 times at bat gives an expected number of six hits in the next 20 attempts.

You Try It #2: The slugging average of a baseball player is the total number of bases he gains divided by his official number of times at bat. If a good slugger has a slugging average of .650, how many bases would you expect him to total for his team in his next 40 official times at bat?

Example #3: We have all received sweepstakes entry forms in the mail with these words or similar ones printed in large letters on the front: YOU COULD POSSIBLY HAVE ALREADY WON $1,000,000. JUST FILL IN THE FORM, SEND THE ENTRY IN, AND THE RESULTS WILL BE ANNOUNCED LATER. The small print at the bottom of the back page says:

> one chance in approximately 32,000,000 to win the GRAND PRIZE of $1,000,000;
> one chance in approximately 320,000 to win each of the 10 first prizes of $10,000;
> one chance in approximately 32,000 to win each of the 10 second prizes of $1,000;
> one chance in approximately 3,200 to win each of the 100 third prizes of $100;
> one chance in approximately 320 to win each of the 1,000 gift certificates worth $10.

What can we realistically expect to happen if we send in our entry form? Are we likely to win a large amount of money or even one of the small prizes or will we likely never win anything?

Solution: As we have read in the fine print, our chance of winning the grand prize of $1,000,000 is about one in 32,000,000, so we can expect an average gain of (1/32,000,000)($1,000,000), which is approximately 3.1¢.

Similarly, we have a one in 320,000 chance of winning $10,000, so we expect to gain on the average (1/320,000)($10,000), which is again about 3.1¢. We have one chance in 32,000 to win $1,000, so we expect to gain (1/32,000)($1,000) on the average, which is about 3.1¢, and there is one chance in 3,200 to win $100, which gives an average expected gain of (1/3,200)($100), or about 3.1¢. Also, we have one chance in 320 to win a gift certificate worth $10, so we can expect an average gain of (1/320)($10), which again is approximately 3.1¢.

Adding these values we can expect to win 15.5¢ on the average if we enter the sweepstakes. Sounds great, but we must remember that we have to mail our entry and the stamp will cost 22¢. Therefore our net gain is 15.5¢ minus the 22¢ it costs to enter, or −6.5¢. In other words, we expect to lose 6.5¢ each time we enter the sweepstakes. This is obviously a lot different than winning a prize of $1,000,000.

You Try It #3: Suppose that you pay $1 for one of 1,000 raffle tickets, each having an equal chance to win a prize of an $800 stereo system. What is your expected gain (or loss) for this ticket?

Early information released on the proposed Iowa state lottery said that each purchaser of a $1 lottery ticket would have approximately one chance in nine of instantly winning either $2 or $5, and approximately one chance in 1,000 of winning a $100 prize in a later drawing. The Lottery Commission expected to pay out about one half of the money received in ticket sales as prize money [Daubenmier 1985].

Example #4: What can we expect to gain for each ticket we purchase in the proposed Iowa state lottery game?

Solution: In each case if we multiply the fraction which represents the probability of winning a particular prize times the value of the prize, then we get the expected value for that prize. Adding all these values gives the total expected value for the lottery game.

In this game, each entry has a 1/9 chance of instantly winning either $2 or $5, so we can represent this by the expression (1/18)($2+$5), since half of 1/9 is 1/18 and since the instant winner will win $2 half of the time and $5 half of the time. Also, there is a one in 1,000 chance of winning $100 later, so the total expected value of each entry is (1/18)($2+$5) + (1/1,000)($100). Calculated and rounded to the nearest cent, this is 49¢, so every time we pay $1 to purchase a lottery ticket we expect a return of 49¢. In the long run, we can expect to lose 51¢ every time we play!

You Try It #4: What is the expected loss for each ticket purchased if the instant winners receive either $4 or $8?

You Try It #5: A fair game is one in which the expected return is equal to the cost of entering; in other words, the expected gain or expected loss is zero. If the instant winners get either $6.20 or $10, is the lottery a fair game?

More Expectation: Fast Servers in Tennis

Most of us have played tennis at some time and realize the difficulty in making a good, strong service. The tennis rule makers realize this too and allow a second service attempt if the first is not good. Suppose that Ted Sinnet has a strong first serve which, when good, wins him 75% of the points, but it is only good one half of the time. His second serve is good 3/4 of the time, but he wins only 50% of the subsequent points. We can use our ideas of expected value to calculate the percentage of points Ted wins when he is serving, remembering that when he fails on both serves he loses the point.

A tree diagram is useful to illustrate this problem, as shown in Figure 1.

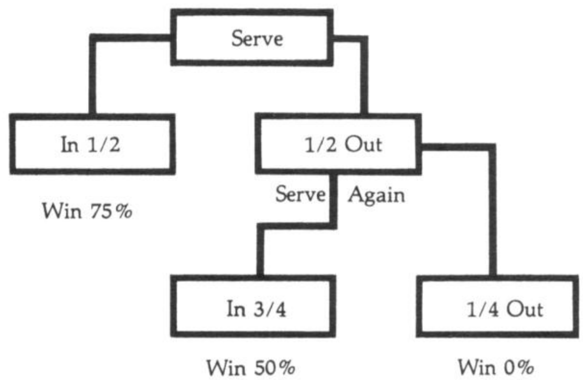


Figure 1.

Example #5: Calculate the percentage of points that Ted Sinnet expects to win when he is serving.

Solution: The percentage of points Ted expects to win when he serves is calculated by multiplying the probability for each case by the percentage of points won in each case and adding to find the expected value.

$$(1/2)(75\%) + (1/2)(3/4)(50\%) + (1/2)(1/4)(0\%) = 56.25\%.$$

You Try It #6: If Sam Lang's serves are all the same, getting six of 10 in play but winning only 60% of the points on each service, what percent of the total points he serves does he expect to win?

You Try It #7: Suppose that Steve Snell has a tremendously powerful first serve with which he wins the point 80% of the times he gets it in play, but he only succeeds in getting it in play one out of three times. His second service is good three of five tries but he only wins 40% of the subsequent points. What percent of the total points he serves does he expect to win?

You Try It #8: If Avery Age hits both serves the same, getting 65% of each service in play and winning 65% of the points, which of the four players, Ted Sinnet, Sam Lang, Steve Snell, or Avery Age, is expected to win the highest percentage of points served?

Extending Expectation: Pitching and Batting

An extension of this idea is a simplified version of the confrontation between the pitcher and batter in a baseball game. Suppose that Pete is pitching and can throw a fastball and a curveball such that 3/4 of his fastballs are in the strike zone but only 2/5 of his curves are in the strike zone. Bill, who is batting, reaches base 13.33% of the time versus each fastball in the strike zone (remember he gets up to three strikes), but reaches base only 5% of the time versus curveballs in the strike zone. Assume that Bill swings only at a pitch in the strike zone and he never swings at a bad pitch (a ball). We shall ignore the possibility of foul balls.

Use a tree diagram to illustrate this baseball simulation. The beginning of one is shown in Figure 2.

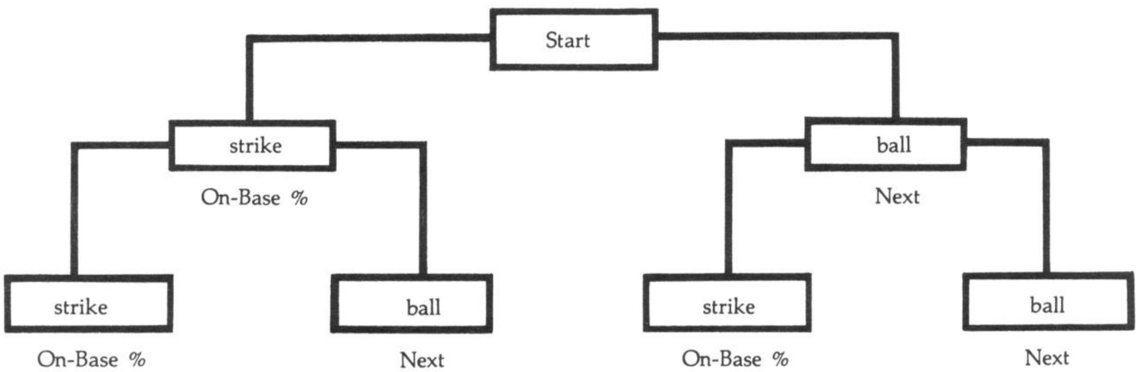


Figure 2.

Example #6: Suppose that Pete pitches only fastballs. What percent of the times that Bill comes to bat would you expect the situation to occur that he reaches base on the pitch following a no-ball, two-strike count?

Solution: Consider this situation one pitch at a time. Pitch 1 is in the strike zone and Bill does not reach base. Pitch 2 is in the strike zone and Bill does not reach base again. Pitch 3 is in the strike zone and Bill does reach base. The probability of a pitch in the strike zone is 3/4 because Pete is throwing only fastballs and the probability of Bill reaching base is .1333 because he reaches base versus 13.33% of fastballs. Therefore, the probability of Bill not reaching base is .8667, since $1 - .1333 = .8667$.
So the probability of this situation occurring is:

(prob. on pitch 1) (prob. on pitch 2) (prob. on pitch 3) =
$(.75)*(.8667)*(.75)*(.8667)*(.75)*(.1333)$ = about 4.22%.

You Try It #9: Suppose that Pete pitches only fastballs. What percent of the times Bill comes to bat would you expect the situation to occur that he reaches base on the pitch following a one-ball, one-strike count?

You Try It #10: Again suppose that Pete is pitching only fastballs. Construct the full tree diagram of 35 branches to help find the percent of times Bill comes to bat that you would expect the situation to occur that he reaches base on a hit following a three-ball, one-strike count.

Example #7: Suppose that Pete pitches only curveballs. What percent of the times Bill comes to bat would you expect the situation to occur that he reaches base on a hit following a full (three-ball, two-strike) count?

Solution: You can see on the complete tree diagram (in the solution to **You Try It #10**) that there are 10 possible cases in which Pete throws a strike for Bill to attempt to hit following a full count. Again using the representation of S equaling the probability of a strike being thrown, B the probability of a ball where $B = 1 - S$, H the probability of a hit, and $1 - H$ the probability of Bill not getting a hit, the solution is $10*B*B*B*S*S*S*(1-H)*(1-H)*H$, or about 0.62% of the time.

You Try It #11: Suppose that Pete pitches only curveballs. What percent of the time that Bill comes to bat would you expect the situation to occur that he reaches base on a hit following a two-ball, two-strike count?

As you can see from the tree diagram, there are 35 possible branches, each with a significant computation involved, in any particular complete solution of this baseball simulation problem. The following computer program totals each of the branches to output the percent of times Bill reaches base under given conditions.

```
10 REM BASEBALL SIMULATION
20 REM H = PERCENT OF HITS FROM EACH
   STRIKE THROWN
30 REM S = PERCENT OF STRIKES THROWN
40 REM B = 100 - S = PERCENT OF BALLS
   THROWN
50 INPUT "WHAT IS THE PERCENT OF HITS
   FROM EACH STRIKE?";H
60 INPUT "WHAT IS THE PERCENT OF
   STRIKES THROWN?";S
70 H = H/100
80 S = S/100
90 B = 1 - S
100 X1 = H*S*(1+B+B*B+B*B*B)
110 X2 = H*(1-H)*S*S*(1+2*B+3*B*B+4*
    B*B*B)
120 X3 = H*(1-H)*(1-H) * S*S*S *
    (1+3*B+6*B*B+10*B*B*B)
130 X4 = B*B*B*B * (1+4*S*(1-H) +
    10*S*S*(1-H)*(1-H))
140 P = X1 + X2 + X3 + X4
150 PRINT "THE PERCENT OF TIMES BILL
    REACHES BASE IS":P*100
160 END
```

Line 110 in the program is the sum, put in the location labelled X2, of the percent of times when Bill possibly reaches base on the second strike thrown to him. We must multiply H by $1-H$ (the probability that he does not reach base on the first strike) times $S*S$, the probability of Pete throwing two strikes, times the probability of Pete throwing balls on any given pitch. These situations occur once when Bill has no balls, 2 times with one ball, 3 times with two balls, and 4 times with three balls.

You Try It #12: What percent of the time can Bill expect to reach base if Pete pitches all fastballs?

You Try It #13: What percent of the time can Bill expect to reach base if Pete pitches all curveballs?

We have found applications of expected value in weather prediction, raffles, sweepstakes, lotteries, tennis, and baseball. There are many other uses of expected value which you will encounter on the radio and television or in newspapers, books, and magazines.

I would like to thank Prof. Hal Schoen for his helpful suggestions and comments on this paper.

References

Daubenmier, Judy. 1985. Iowa lottery. *Cedar Rapids Gazette.* Cedar Rapids, IA, 7.

Travers, K., W. Stout, J. Swift, and J. Sextro. 1985. *Using Statistics.* Menlo Park, CA: Addison-Wesley Publishing Co.

Solutions to "You Try It"

1. (20)(60%) = 12. You would expect it to rain on 12 of these 20 days.

2. (.650)(40) = 26. You expect him to total 26 bases in these 40 times at bat. For example, he might get three singles (three total bases), four doubles (eight total bases), one triple (three total bases), and three home runs (12 total bases); these hits yield 3+8+3+12 = 26 total bases.

3. (1/1000)($800) − $1 = −$0.20. You would expect to lose 20¢ on each ticket.

4. (1/18)($4+$8) + (1/1000)($100) − $1 = −$0.23. You would expect to lose 23¢ for each lottery ticket purchased.

5. (1/18)($6.20+$10) + (1/1000)($100) − $1 = $0.00. Yes, this would be a fair game.

6. (6/10)(60%) + (4/10)(6/10)(60%) + (4/10)(4/10)(0%) = 50.4%. Sam Lang would expect to win about 50% of the points when he serves.

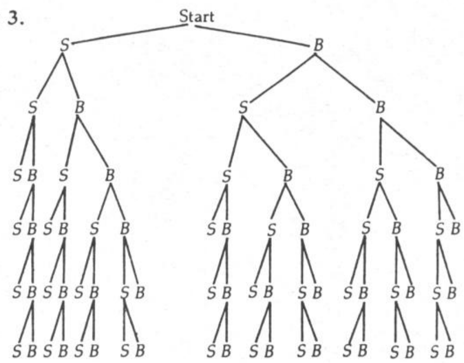
7. (1/3)(80%) + (2/3)(3/5)(40%) + (2/3)(2/5)(0%) = 42.67%. Steve Snell would expect to win about 43% of the points when he serves.

8. (65/100)(65%) + (35/100)(65/100)(65%) + (35/100)(35/100)(0%) = 57.04%. Avery Age has the highest expected percentage of service points won, about 57%.

9. Let S equal the probability of throwing a strike and $B = 1-S$ equal the probability of throwing a ball. If H is the percent of times Bill gets a hit when a strike is thrown, then $1-H$ is the percent of the time that Bill does not get a hit when a strike is thrown. In this problem $S=.75$, $B=.25$, $H=.1333$, and $1-H=.8667$. There are two ways a one-ball, one-strike count could occur. Either Pete throws a strike (which Bill does not hit) and then a ball, or Pete throws a ball and then a strike (which Bill does not hit). Multiply each of these probabilities by $S*H$, which is the probability of the next pitch being in the strike zone and Bill getting a hit. Thus the solution is $S*(1-H)*B*S*H +$ $B*S*(1-H)*S*H +$ $2*B*S*S*(1-H)*H$, about 3.25%.

10. The complete tree diagram is shown in Figure 3. You find four cases of a three-ball, one-strike count, which gives $4*B*B*B*S*(1-H)*S*H =$ $4*B*B*B*S*S*(1-H)*H$, about 0.41%.

Figure 3.



11. Referring again to the complete tree diagram, you find six cases of a two-ball, two-strike count for which Bill does not hit either of the two strikes, so the solution is $6*B*B*S*S*S*(1-H)*(1-H)*H$ with $S=.4$, $B=.6$, and $H=.05$, since Pete is pitching only curveballs. This is about 0.62%.

12. Using the computer program to total the percent of the time Bill reaches base with 13.33%, the percent of hits from each fastball thrown, and 75%, the percent of fastballs Pete can throw for strikes, Bill reaches base about 37.34% of the time against all fastballs.

13. Using the same program with 5%, the percent of hits from each curveball thrown, and 40%, the percent of curveballs Pete can throw for strikes, Bill reaches base about 60.93% of the time against all curveballs, some of this coming from Pete's inability to throw strikes with a curveball.