# Voting: The Importance of Method

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When people put an issue to a vote, they generally suppose that they are using a fair and accurate way of discovering what the group prefers. Occasionally, they are concerned about possible dishonesty or voter apathy, but they seldom ask whether their voting method may be flawed. This is a serious omission, for a voting method may select a choice that is not what the voters want most. This Pull-Out Section points out the weaknesses in the sequential and plurality voting methods and proposes an alternative voting process call the Black method.

Sequential Pairwise Voting

The process called *sequential pairwise voting* is used when there are more than two choices. First, one pair of decision options is considered and a vote is taken; then, the winner is paired with another option for another vote. This method of pairing and voting continues until a final winner is selected. A flaw in this method is illustrated by the following example.

Each year, to honor the person who has contributed the most to the council, the officers of the Student Council at Central High School select the "Member of the Year." This year, there is a lot of concern about the selection, for it is rumored that last year's officers selected poorly. That rumor is true. Last year's selection was David Kovacs. David was not unworthy but, after the choice had been announced, it was learned that each of the three officers (president, vice president, and secretary-treasurer) preferred Barbara Ostrowski to David Kovacs. (See **Figure 1**.)
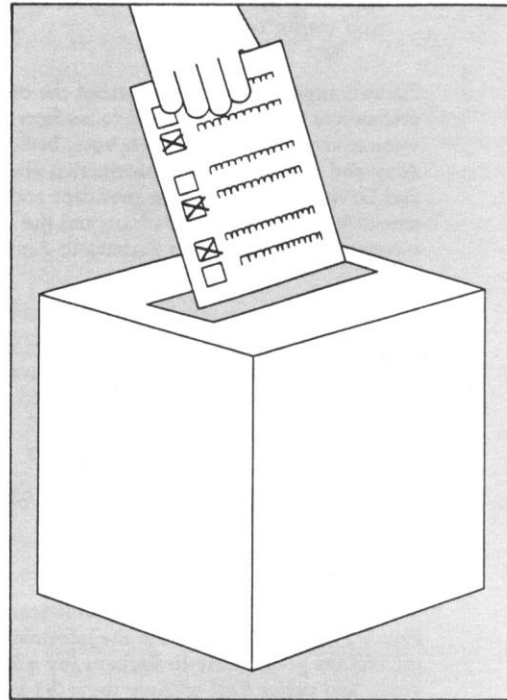
Here is the story behind the selection:

Last year, nominations for "Member of the Year" were submitted in writing by Student Council members to the officers. They were slow to come in. At first the only two nominees were Amy Snyder and Barbara Ostrowski. By a 2-1 vote, the officers selected Amy. Before it was time to announce the selection, Carl Evans was nominated. The officers saw no need to vote on Barbara again. They considered Amy and Carl and, by a 2-1 vote, selected Carl. Just after this voting, a fourth nomination was made, David Kovacs. In the choice between Carl and David, David won with a 2-1 vote.

Figure 1 supplies information about the officers' preferences and it can be used to see how the vote went at each step. For the first vote, between Amy and Barbara, ignore information about Carl and David; observe that the president and vice-president prefer Amy to Barbara and the secretary-treasurer prefers Barbara to Amy.

	President	Vice-President	Secretary-Treasurer
1st choice	Amy	Carl	Barbara
2nd choice	Barbara	Amy	David
3rd choice	David	Barbara	Carl
4th choice	Carl	David	Amy

Figure 1. Officers' rankings of "Member of the Year" nominees.

You Try It #1: Confirm your understanding of Figure 1 by using it to obtain the information that the officers prefer Amy to Barbara (by a 2-1 vote), and prefer Carl to Amy (by a 2-1 vote) and prefer David to Carl (by a 2-1 vote).

The arrangement for voting first between Amy and Barbara, then between Barbara and Carl, and finally between Carl and David may be described as an *agenda*. This agenda and its consequences are diagrammed in Figure 2.
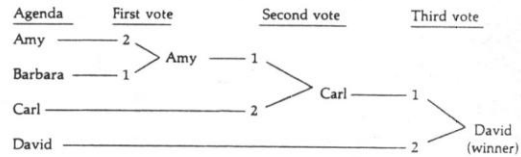

Figure 2. The results of the Amy-Barbara-Carl-David agenda.

To see the effect of a different agenda on the outcome, suppose that the nominations had come in reverse order, so that first David was paired with Carl, then the winner was paired with Barbara, and the winner of that pairing was paired with Amy. Figure 3 shows Amy to be the winner in that case.
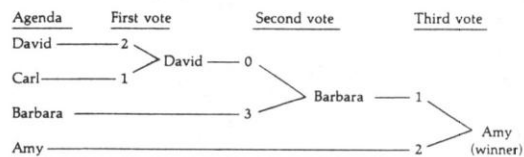

Figure 3. The results of the David-Carl-Barbara-Amy agenda.

You Try It #2: Experiment with different agendas for selection of student council "Member of the Year" and find one that leads to Carl's selection. Refer to Figure 1 for information about how the officers will vote; diagram the agenda and the voting as in Figures 2 and 3.

You Try It #3: Find an agenda that leads to Barbara's selection and diagram the agenda and voting as in Figures 2 and 3.

You Try It #4: Tell whether you agree or disagree with the following statement and supply reasons:

> When sequential pairwise voting is used, the agenda rather than the voters, makes the decision.

You Try It #5: (a) There are 24 possible agenda orders for voting to select one of the four nominees for "Member of the Year." Diagram these agendas and determine the winner in each case.

(b) For how many of the 24 agendas in (a) did the first person on the agenda emerge as the winner? For how many agendas did the second person emerge as the winner? The third? The fourth?

You Try It #6: The power of the agenda to influence the outcome of an election when sequential pairwise voting is used is called the *agenda effect*. The following "rule-of-thumb" is proposed as a summary of the agenda effect on sequential pairwise voting:

> Regardless of voter opinions, the later an alternative's position on the agenda, the better its chances of selection as the winning alternative.

Use the results of You Try It #5 to evaluate this rule-of-thumb.

You Try It #7: Consider an election in which there are five voters and three candidates (Peters, Quincy, and Roudebush) with the voters' rankings given in Figure 4. Examine the agenda effect on this case.

	Voter 1	Voter 2	Voter 3	Voter 4	Voter 5
First choice	P	P	Q	Q	R
Second choice	R	R	R	R	Q
Third choice	Q	Q	P	P	P

Figure 4. Voters' rankings of Peters, Quincy, and Roudebush.

You Try It #8: Test your understanding of the agenda effect by creating an election with five voters deciding among three choices L, M, and N in which each choice can be a winner if an appropriate agenda is used.

Choosing a Voting Method that Avoids the Agenda Effect

A common voting method is the *plurality method*, in which all choices are considered at once and each voter casts a single vote for the choice he or she most prefers.

You Try It #9: With 1000 voters and the plurality method of voting, what is the smallest fraction of the voters that is sufficient to elect a candidate if there are three candidates? Four candidates? Five candidates? Ten candidates? Comment on the desirability of the plurality method in such cases.

You Try It #10: (**a**) What would have been the outcome if the student council officers, whose preferences are given in Figure 1, had used plurality voting?

(**b**) What would have been the outcome if the voters whose preferences are given in Figure 4 had used plurality voting?

You Try It #9 and You Try It #10 illustrate why the plurality method is not a wholly satisfactory way of selecting an election winner. Although You Try It #9 points out that the plurality winner may be selected by a small fraction of the voters (just over $1/n$ of the voters in an election with n choices) this is not the only cause for concern. More important is that the plurality method ignores information about the second and third (and so on) choices of the voters—information that is useful in evaluating their consensus. In You Try It #10 one finds that the plurality method fails to select winners in the elections of Figures 1 and 4 even though the voters' preference rankings supply enough information to select preferred candidates.

Although neither the plurality method nor the sequential pairwise method always fails to give a fair and accurate assessment of voter consensus, we cannot know in advance whether they will fail. Thus it is sensible to avoid these methods, if possible. There is a cost, however; voters must be willing to take more trouble with an election.

The Marquis de Condorcet, an eighteenth century mathematician, philosopher, and political leader, proposed the following means of deciding elections:

> Each pair of alternatives should be considered in its own separate election and the winner determined. If one alternative emerges as winner over all the others in these separate two-way contests, then that alternative is the voters' preferred choice.

If we apply Condorcet's method to the election described in Figure 4 of You Try It #7, we have the results shown in Figure 5.

Election between P and Q

   P: 2 votes
   Q: 3 votes (winner)

Election between P and R

   P: 2 votes
   R: 3 votes (winner)

Election between Q and R

   Q: 2 votes
   R: 3 votes (winner)

Figure 5. Two-way contests among candidates P, Q, and R.

Condorcet's method selects candidate R as the winner, since R beats each of P and Q in two-way contests. The results of all two-way contests may be conveniently displayed in a diagram called an *election digraph*. Each candidate is represented by a dot or *vertex* of the diagram and each two-way contest is represented by an arrow or *directed edge* from the winner's vertex to the loser's vertex. Each end of the directed edge is labeled with the number of votes that each of the candidates received. An election digraph for the results shown in Figure 5 is given in Figure 6.
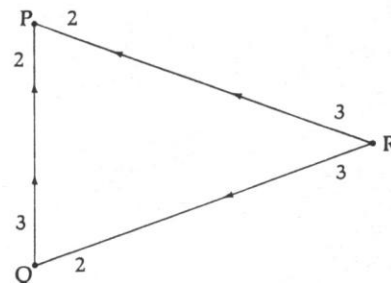
Figure 6. Election digraph for two-way contests among P, Q, and R.

You Try It #11: Consider Condorcet's method for deciding elections. What are its strengths and weaknesses?

Applying Condorcet's method to the election of student council "Member of the Year" requires us to consider six two-way contests. The results of these are shown in Figure 7 and the related election digraph is shown in Figure 8.

Election between **A** and **B**

> **A:** 2 votes (winner)
> **B:** 1 vote

Election between **A** and **C**

> **A:** 1 vote
> **C:** 2 votes (winner)

Election between **A** and **D**

> **A:** 2 votes (winner)
> **D:** 1 vote

Election between **B** and **C**

> **B:** 2 votes (winner)
> **C:** 1 vote

Election between **B** and **D**

> **B:** 3 votes (winner)
> **D:** 0 votes

Election between **C** and **D**

> **C:** 1 vote
> **D:** 2 votes (winner)

Figure 7. Two-way contests among candidates A, B, C, and D.
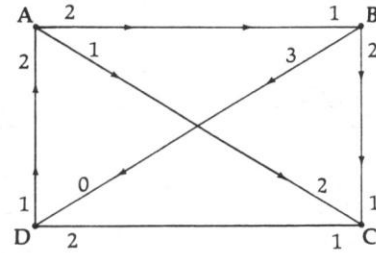
Figure 8. Election digraph for two-way contests among **A**, **B**, **C**, and **D**.

Because none of the candidates **A**, **B**, **C**, or **D** is able to beat all of the others in two-way elections, Condorcet's method does not select a winner for the "Member of the Year" election. To supplement Condorcet's method in a case when it does not find a winner, the twentieth century British political scientist, Duncan Black, has proposed a method which offers a back-up procedure. Black's method follows the Condorcet method if it selects a winner; if it does not, the total number of votes from all two-way contests should be counted, and the winner should be that candidate whose total is greatest.

Black's method can be applied to the "Member of the Year" election by referring to the election digraph shown in Figure 8. Summing the numbers around the vertex for each candidate gives the total number of votes received in two-way contests. These results, shown in Figure 9, select **B** as winner.

Candidate	Total of two-way contest votes
A	5
B	6
C	4
D	3

Figure 9. Vote totals from all two-way contests in "Member of the Year" election.

Sequential pairwise voting allowed the possibility that an electorate might select a choice that was less preferred by all voters than an unselected choice. Black's method avoids this difficulty and thus is a superior method. When compared with the plurality method, however, the superiority of Black's method is less clear. Black's method utilizes more information from voters and requires more effort. This characteristic is a strength of the method and yet it may also be a weakness. In small elections in which voters are well informed and care a lot about the outcome, the effort required by Black's method is likely to be willingly given. But in large elections in which voters are poorly informed and tend to be apathetic, a complex voting method may cause voters to be careless or to refrain from voting at all.

Black's method is not, even for small elections, a completely perfect voting method. In fact, investigations pioneered by Kenneth Arrow, Nobel prize-winning economist, and followed up by others, have shown that there does not exist any voting method that satisfies all conditions on a short list of reasonable properties that one might wish for a democratic voting method to satisfy. [See HiMAP Module 1.]

You Try It #12: Apply the Black method to determine the winner for each of the following elections. Display the election results using an election digraph. In each case, comment on the result. Can you find a reason why a candidate other than the one chosen by the Black method should have won? Or is the winner, in your view, the preferred choice of the group?

Note: We have simplified the rankings in elections below by grouping all voters with like preferences under a single heading. One should be careful to remember that these elections have 9 or 11 voters, not 3 or 4.

(a) ELECTION 1 (9 voters)

4 voters	3 voters	2 voters
B	O	G
G	G	O
O	B	B

(b) ELECTION 2 (11 voters)

2 voters	3 voters	3 voters	3 voters
F	F	G	H
G	E	E	E
H	G	F	G
E	H	H	F

(c) ELECTION 3 (9 voters)

4 voters	3 voters	2 voters
E	G	F
F	E	G
G	F	E

You Try It #13: Apply the Black method to the election that you created in Exercise B. Do you approve of the result?

You Try It #14: An Application of Voting to Personal Decision Making.

Raymond is a senior at Euphoria College, majoring in accounting. He is trying to decide which job to take after graduation. After many applications and several interviews, there are four job offers that he is seriously considering. The criteria that he considers important are location, salary, and opportunity for advancement. The lists in Figure 10 show his rankings; each job is identified by the city in which it is located.

	Location	Salary	Opportunity for Advancement
Best	Chicago	Atlanta	Birmingham
Second best	Atlanta	Birmingham	Dallas
Third best	Birmingham	Dallas	Chicago
Fourth best	Dallas	Chicago	Atlanta

Figure 10. Raymond's rankings of job offers.

Raymond considers salary and opportunity for advancement to be equally important and both of these criteria to be twice as important as location. He thus decides to treat the problem as an election with voter preferences as shown in Figure 11.

	Location	Salary		Opp. for Adv.	
	Voter 1	Voter 2	Voter 3	Voter 4	Voter 5
First choice	C	A	A	B	B
Second choice	A	B	B	D	D
Third choice	B	D	D	C	C
Fourth choice	D	C	C	A	A

Figure 11. An election that represents Raymond's job decision.

(a) Apply the Black method to Raymond's decision election. Does it select the choice that you think he prefers? Explain.

(b) Suppose that Raymond's job offers weren't all available at once. First offers from **A** and **B** arrived. He chose the one he preferred and said "No" to the other. But before he had signed a contract with his preferred choice, an offer came from **C**. He compared **C** with his preferred choice (**A** or **B**), identified which he now preferred, and said "No" to the other. But before he had signed a contract with this preference, an offer came from **D**. He compared **D** with his earlier choice and made his final decision. Diagram the agenda of Raymond's choice and show how he was a victim of the agenda effect.

(c) Think of an example from your own experience in which, as in part (b), the timing of decision choices forced the use of sequential pairwise voting. In the example, did agenda order lead to a decision that was not the most preferred choice overall?

This Pull-Out Section has merely scratched the surface of a large problem, the selection of a voting method that fairly and accurately reflects the consensus of the voters. Newspapers, magazines, and history books provide examples of elections in which the person elected seems not to have been the voters' choice. Your own reading of history and current events probably has already supplied you with examples of elections in which the results seem not to have reflected the voters' preferences. The following references provide places to start further investigation.

References and Suggestions for Further Reading

Brams, Steven J. and Peter C. Fishburn, *Approval Voting*, Birkhauser, Boston, 1983.

Farquharson, Robin, *Theory of Voting*, Yale University Press, New Haven, CT, 1969.

Growney, JoAnne Simpson, *Mathematics in Daily Life: Making Decisions and Solving Problems*, McGraw-Hill, New York, 1986.

Malkevitch, Joseph, *The Mathematical Theory of Elections*, HiMAP Model 1, Consortium for Mathematics and Its Applications, 1985.

Malkevitch, Joseph and Walter Meyer, *Graphs, Models and Finite Mathematics*, Prentice-Hall, Englewood Cliffs, NJ, 1974.

Straffin, Philip D. Jr., *Topics in the Theory of Voting*, COMAP, Inc., 1980.

The World Almanac and Books of Facts, Newspaper Enterprise Association, New York, any year's edition.

Solutions to "You Try It"

2. The agenda D-A-B-C is one that will result in Carl as winner.

3. The agenda A-C-D-B will result in Barbara as winner.

4. For the "Member of the Year" election, the given statement is true. However, not so for all elections. (See, for example, the election of Figure 4 and You Try It 7.)

5. (a) Of the 24 possible agendas, 10 select Amy (6 with Amy last on the agenda and 4 with her next-to-last), 6 select Barbara (4 with her last, 2 with her next-to-last), 6 select Carl (all with him last), 2 select David (both with him last).

   (b) For 18 of the 24 possible agendas, the fourth and last one on the agenda is selected; in the other 6 agendas the third (next-to-last) is selected.

6. For the "Member of the Year" election this statement is true. But a single example doesn't provide a complete good test for the rule, does it?

7. All agendas lead to the selection of R as winner.

8. One possible example:

	1 Voter	1 Voter	1 Voter	1 Voter	1 Voter
First choice	K	K	L	L	M
Second choice	L	L	M	M	K
Third choice	M	M	K	K	L

   A question to consider: Suppose the 5th voter (who prefers M) were in charge of setting the agenda. Might that person be tempted to choose an agenda that will cause M to win? (More generally, might *manipulation of the agenda* be still another drawback to sequential pairwise voting?)

9. 334/1000, 251/1000, 201/1000, 101/1000
   The plurality method could lead to the undesirable result of selection of a winner that only a small fraction of the voters prefer.

10. (a) The results of plurality voting:
       Amy: 1 vote
       Barbara: 1 vote
       Carl: 1 vote
       David: 0 votes

   (b) The results of plurality voting:
       P: 2 votes
       Q: 2 votes
       R: 1 vote

Comment: The primary drawback of the plurality method in these examples is not that ties have occurred since this can happen with any election method. (In public elections it is customary to resolve ties by random methods, such as tossing a coin or drawing a number.) A primary difficulty of the plurality method in these cases is that it ignores information from the voters that could be used to determine which candidate they really prefer.

11. A primary *strength* of Condorcet's method is that it prevents the possibility of election of a candidate that is not preferred to all others. A *weakness* is that the method may fail to select a winner. (See, for example, discussion pertaining to Figures 7 and 8.) In large elections, the complexity of the method—as compared with the simplicity of the plurality method, for example—may also be considered a weakness.

12. (a) In Election 1 the Black method selects G, who can defeat each of O and B in two-way contests.

   (b) In Election 2 the Black method selects E, who can defeat each of F, G, and H in two-way contests.

   (c) The Black method selects E.
       In two-way contests E beats F (7-2), F beats G (6-3), and G beats E (5-4); thus the Condorcet method does not select a winner. Summing the votes earned in two-way elections gives:

       E: 11 votes (winner)
       F: 8 votes
       G: 8 votes

14. (a) The Black method selects Birmingham as Raymond's preferred choice.
       In two-way contests A beats B (3-2), C beats A (3-2), A beats D (3-2), B beats D (5-0), B beats C (4-1), and D beats C (4-1); thus the Condorcet method does not select a winner.
       Summing the votes earned in two-way elections gives:

       Atlanta: 8 votes
       Birmingham: 11 votes (winner)
       Chicago: 5 votes
       Dallas: 6 votes

   (b) With the agenda A-B-C-D and sequential pairwise voting, D will be selected. Thus Raymond would indeed be a "victim" of the agenda effect, for all of his rankings list B above D.