HiMAP Pull-Out Section: Winter 1990

Card Shuffling, Card Tricks, and Mathematics

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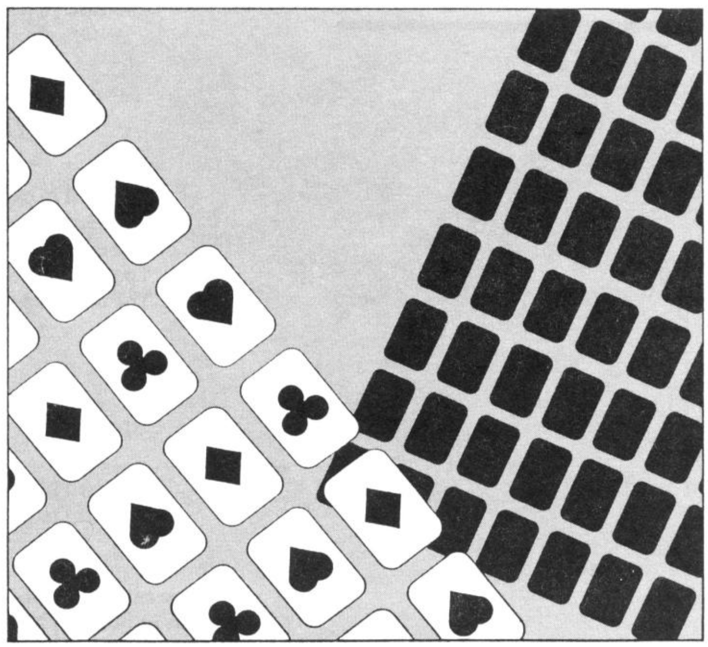

References

Crouse, R.J., and C. Reese. 1973. Using Algebra to Solve an Interesting Card Trick. *Mathematics Teacher*. 67, 653-654.

Kolata, G. 1982. Perfect Shuffles and their Relation to Math. *Science*. 216, 505-506.

Ms. Schwartzman, one of the co-authors and a high school mathematics teacher, suggests that the teacher review modular arithmetic with the students as preparation for doing this HiMAP Pull-out Section. This material is from a talk given by the authors at the October, 1989, NCTM Conference at Denver, Colorado.

Professor Peter Diaconis of Harvard has discovered that seven shuffles are required to mix a deck of 52 cards thoroughly. Rather surprisingly, after eight perfect shuffles, the order of the cards in a deck is identical to the original order. Let us see why this is so.

By a *perfect shuffle* we mean one in which the deck is first cut into two equal halves (each containing 26 cards) and then the two halves are meshed together so that, if the original sequence of cards was c_0, c_1, c_2, . . ., c_{50}, c_{51}, then the new sequence of cards is c_0, c_{26}, c_1, c_{27}, c_2, c_{28}, . . ., c_{25}, c_{51}. Notice that cards c_0 and c_{51} remain in their original positions. Therefore, we need only keep track of cards c_1, c_2, . . ., c_{50}. If we refer to the original location of card c_1 as *position 1*, that of card c_2 as *position 2*, and so on, we see that the following changes take place:

The card in position 1 moves to position 2.
The card in position 2 moves to position 4.
The card in position 3 moves to position 6.

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The card in position 24 moves to position 48.
The card in position 25 moves to position 50.
The card in position 26 moves to position 1.
The card in position 27 moves to position 3.
The card in position 28 moves to position 5.

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The card in position 49 moves to position 47.
The card in position 50 moves to position 49.

We can state that, after one shuffle, the card initially in position q will be in position $2q \pmod{51}$. This is true since $2(1) \equiv 2 \pmod{51}$, $2(2) \equiv 4 \pmod{51}$, . . ., $2(25) \equiv 50 \pmod{51}$, $2(26) \equiv 1 \pmod{51}$, $2(27) \equiv 3 \pmod{51}$, $2(28) \equiv 5 \pmod{51}$, . . ., $2(50) \equiv 49 \pmod{51}$. Therefore, after k shuffles, the card initially in position q will be in position $2^k q \pmod{51}$. The order of the cards is identical to the original order after k shuffles if and only if $2^k q \equiv q \pmod{51}$ for $q = 1, 2, \ldots, 50$. In particular, the order of the cards is identical to the original order after k shuffles if and only if $2^k \equiv 1 \pmod{51}$. Since $2^8 \equiv 1 \pmod{51}$, the order of the cards after eight shuffles is identical to the original order.

We can generalize this result to a deck containing an even number of n cards: The order of the cards is identical to the original order after k shuffles if and only if $2^k \equiv 1 \pmod{n-1}$.

Do "You Try Its" #1 – #4.

Interestingly, the above result looks very similar to a theorem about the decimal representation of rational numbers, namely the period of the decimal representation of the reduced fraction *a/b*, where *b* contains no factors of 2 or 5, is the smallest natural number k for which $10^k \equiv 1 \pmod{b}$. For example, the decimal representation of the fraction 10/37 has period 3 since $10^3 \equiv 1 \pmod{37}$. Indeed, $10/37 = 0.\overline{270}$. For base two arithmetic, the analogous theorem is that the period of the representation of the reduced fraction *a/b*, where *b* is odd, is the smallest natural number k for which $2^k \equiv 1 \pmod{b}$. For example, since $2^4 \equiv 1 \pmod{15}$, the period of 1/15 in its base two representation must be 4. Writing $15 = 1111_{\text{base two}}$, the long division at the left gives us the base two representation of 1/15.

$$\begin{array}{r} 0.0001 \\ 1111\,\overline{)\,1.0000} \\ \underline{1111} \\ 1 \end{array}$$

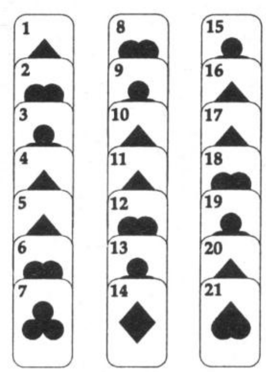
Thus, $1/15 = 0.\overline{0001}_{\text{base two}}$.

Do "You Try Its" #5 and #6.

A Card Trick

In a fairly well-known card trick, the player is allowed to see the cards that appear among three columns of seven layered cards each. He is instructed to think of a card in the columns and to point to the column containing the card. The "magician" folds up the three columns into three stacks, places the indicated stack between the other two, and creates another 7 by 3 array by dealing from the top left to right, then down to the next row left to right, and so on. For example, suppose the array of cards was as in **Figure 1** and the player's card is 6.

Array of Cards After the Initial Placement

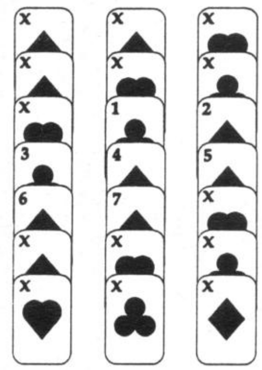


The selected card, the 6, is the second card below the middle card of its column.

Figure 1.

After the first subsequent deal, the cards would appear as in **Figure 2.**

Array of Cards After the First Subsequent Deal

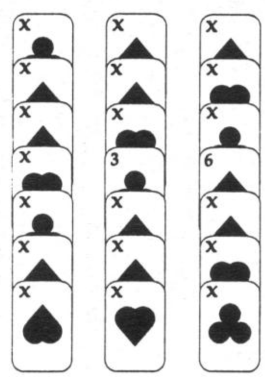


The selected card, the 6, is one card below the middle card of its column.

Figure 2.

Again, the player is asked to point to the column containing his card and, once again, the "magician" folds up the columns into three stacks, places the indicated stack between the other two, and creates another 7 by 3 array by dealing left to right, then down to the next row, and so on. The array would appear as in **Figure 3.**

Array of Cards After the Second Subsequent Deal



The selected card, the 6, is in the middle of its column.

Figure 3.

When the player now points to the third column, the "magician" knows for certain that the player's card is the 6 since the player's card will always be in the middle of the identified column after the second subsequent deal.

This card trick is, of course, mathematically based and suggests some questions worth exploring. First, why does it work and, second, what happens if the number of rows is different from seven?

Why Does the Card Trick Work?

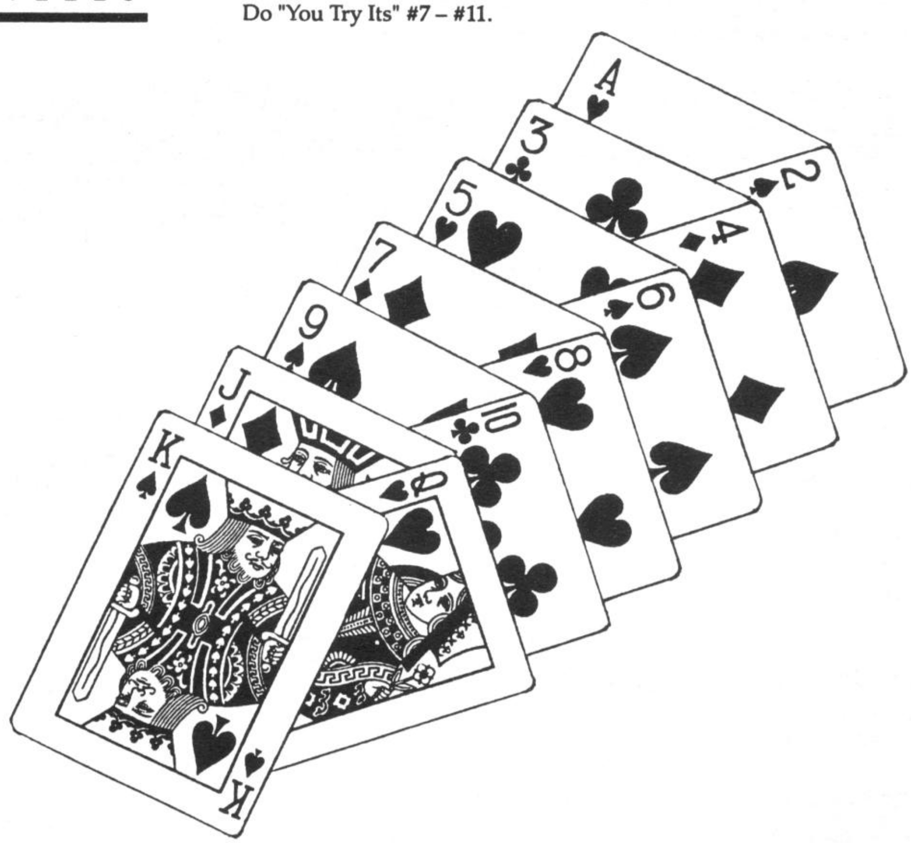
Suppose we have three columns each containing the same odd number of cards. When the "magician" places the indicated stack between the other two, the middle card in the indicated column will be the middle card in the middle column after the next deal. **Table 1** shows how far, after the next deal, the player's card will be from the middle card of its column, given how far it currently is from the middle card of its column.

Table 1
Distance of selected card from the middle card of its column.

Currently	0	1	2	3	4	5	6	7	8
After Next Deal	0	0	1	1	1	2	2	2	3

From this table, it can be seen that, if the player's card currently is d cards from the middle card of its column, then, after the next deal, the player's card will be $[(d + 1)/3]$ cards from the middle card of its column ($[x]$ denotes the greatest integer less than or equal to the number x).

Do "You Try Its" #7 – #11.



You Try It #1

Determine how many perfect shuffles are required for a deck of 22 cards to return to its initial state.

You Try It #2

Write a computer program that determines, for a given even number of cards, the number of perfect shuffles required to return the deck to its initial state.

You Try It #3

Show that a deck of 2^m cards returns to its initial state after m perfect shuffles. Write out an argument.

You Try It #4

Show (write out an argument) that a deck of $2^m + 2$ cards returns to its initial state after $2m$ perfect shuffles.

You Try It #5

a Determine k such that $2^k \equiv 1 \pmod{7}$.

b Use part (a) to find the period of 1/7 in its base two representation.

c Find the base two representation of 1/7.

You Try It #6

a Determine k such that $2^k \equiv 1 \pmod{9}$.

b Use part (a) to find the period of 1/9 in its base two representation.

c Find the base two representation of 1/9.

You Try It #7

Beginning with an array of three columns of 9 cards each, show (write out an argument) that the maximum number of subsequent deals (after the initial placement of the cards) until the player's card is in the middle of its column is equal to 2.

You Try It #8

Beginning with an array of three columns of 11 cards each, show (write out an argument) that the maximum number of subsequent deals until the player's card is in the middle of its column is equal to 3.

You Try It #9

Determine all odd natural numbers s for which the following is true: Beginning with an array of three columns of s cards each, the player's card is certain to be in the middle of its column after 3 subsequent deals. You may want to write and report a short computer program using the greatest integer function to do this.

You Try It #10

Determine all odd natural numbers s for which the following is true: Beginning with an array of three column of s cards each, the player's card is certain to be in the middle of its column after 4 subsequent deals.

You Try It #11

Use "You Try Its" #9 and #10 to state a conjecture.

Answers to the You Try Its

1 Six perfect shuffles are required.

2 This is an example program in BASIC. Yours could be different and still be acceptable.

```
10 INPUT N
20 K = 0
30 J = 1
40 K = K + 1
50 J = 2 * J
60 A = (J – 1)/(N – 1)
70 IF A <> INT (A)
   THEN 40
80 PRINT N, K
```

3 The order of a deck of n cards is identical to the original order after k perfect shuffles if $2^k \equiv 1 \pmod{n-1}$. Since $2^m \equiv 1 \pmod{2^m - 1}$, a deck of 2^m cards returns it to its initial state after m perfect shuffles.

4 Notice that $2^m - 1$ is a divisor of $2^{2m} - 1$. The order of a deck of n cards is identical to the original order after k perfect shuffles if $2^k \equiv 1 \pmod{n-1}$. Since $2^{2m} \equiv 1 \pmod{2^m + 2 - 1}$, a deck of $2^m + 2$ cards returns is to its initial state after 2^m perfect shuffles.

5

a $k = 3$.

b 3.

c $0.\overline{001}_{\text{base two}}$.

6

a $k = 6$.

b 6.

c $0.\overline{000111}_{\text{base two}}$.

7 Initially, the player's card is at most four cards from the middle card of its column. After the first subsequent deal, the player's card will be at most $[(4 + 1)/3] = 1$ card from the middle card of its column.

After the second subsequent deal, the player's card will be at most $[(1 + 1)/3] = 0$ cards from the middle card of its column.

8 Initially, the player's card is at most five cards from the middle card of its column. After the first subsequent deal, the player's card will be at most $[(5 + 1)/3] = 2$ cards from the middle card of its column.

After the second subsequent deal, the player's card will be at most $[(2 + 1)/3] = 1$ cards from the middle card of its column.

After the third subsequent deal, the player's card will be at most $[(1 + 1)/3] = 0$ cards from the middle card of its column.

9 $s = 1, 3, \ldots, 27$.

10 $s = 1, 3, \ldots, 81$.

11 Beginning with an array of three columns of s cards each, where s is an odd positive integer, the maximum number of subsequent deals until the player's card is in the middle of its column is equal to the smallest non-negative integer m for which $s \leq 3^m$.